Fuzzy Refutations for Probability and Multivalued Logics

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ABSTRACT

The deductive apparatus of a fuzzy logic is usually proposed in Hilbert's style by fixing a fuzzy subset of logical axioms and a set of fuzzy inference rules. We sketch a "refutation approach" to fuzzy deduction. In particular, this enables us to face probability logic as a particular fuzzy refutation system in the framework of fuzzy logic. Namely, we propose a refutation system in which the probabilistic theories correspond to the lower envelopes and the complete probabilistic theories correspond to the probabilities. Finally, we apply the concept of fuzzy refutation system to multivalued logic.

KEYWORDS: ***fuzzy logic, approximate reasoning, probability, multivalued logic***

1. INTRODUCTION

As is well known, the usual classical deductive systems are not adequate to face the essential vagueness of human reasoning, and therefore one focus of research in artificial intelligence is the problem of "approximate reasoning." Now, an interesting contribution to this subject is fuzzy logic theory, whose basic principles have been formulated by Zadeh in [16] and successively examined by several other authors. The aim of fuzzy logic is to give a precise meaning to expressions like

"v entails that α holds at least to degree λ,"

where v is a fuzzy subset of formulas (the available fuzzy information), α a formula, and $\lambda \in [0, 1]$ a generalized truth value.

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Received and accepted May 26, 1994.

International Journal of Approximate Reasoning 1994; 11:257–279

655 Avenue of the Americas, New York, NY 10010 0888-613X/94/$7.00

On the other hand, in classical logic very useful refutation techniques exist to prove that a formula α is a consequence of a given system Σ of axioms. The starting point is the equivalence between the two claims.

- Σ entails α,
- $\sim \alpha$ is inconsistent with Σ.

So, in order to extend these techniques to fuzzy logics, in this paper we define a notion of degree of inconsistency of a formula α with respect to a fuzzy set v of formulas in such a way that the following are equivalent:

- v entails that α holds at least to degree λ,
- the degree of inconsistency of $\sim \alpha$ with v is λ.

In such a way we obtain a refutation approach to fuzzy logic.

In particular, this enables us to obtain a refutation system for managing information that is probabilistic in nature. In this system the probabilistic theories correspond to the lower envelopes and the complete probabilistic theories correspond to the probabilities. The formulas we find are perhaps useful for the questions examined by Weichselberger and Pohlmann in [14].

Also, we apply the refutation approach to those fuzzy logics whose semantics is furnished by the class of truth functional valuations of a multivalued logic. Such semantics were extensively examined by Pavelka in [10].

2. FUZZY LOGIC

Denote by U the unit interval $[0, 1]$. Then, given a set S, a *fuzzy subset* of S is any element of the direct power $\mathscr{F}(S) = U^S$, that is, any map $s: S \to U$ from S into U (Zadeh [15]). $\mathscr{F}(S)$ inherits the structure of a complete lattice from U, and it is an extension of the lattice $\mathscr{P}(S)$ of all the subsets of S. Indeed, if we define *crisp a* subset s as one such that $s(x) \in \{0, 1\}$ for every $x \in S$, then we can identify $\mathscr{P}(S)$ with the sublattice $\{0, 1\}^S$ of the crisp fuzzy subsets of S. Namely, we identify any $X \in \mathscr{P}(S)$ with the related characteristic function $\chi_X \in \mathscr{F}(S)$. We extend to the lattice $\mathscr{F}(S)$ the terminology of set theory; for example, given two fuzzy subsets s and s' of S, if $s' \leq s$, we say that s' is *included* in s or that s' is a *part* of s, and we write $s \subseteq s'$. Also, we define the *union* $s \cup s'$ and *intersection* $s \cap s'$ as the join and the meet of s and s', that is, the fuzzy subsets defined by setting

$$(s \cup s')(x) = \max\{s(x), s'(x)\} \quad \text{and} \quad (s \cap s')(x) = \min\{s(x), s'(x)\} \tag{2.1}$$

for every $x \in S$. Likewise, one defines the *union* $\bigcup s_i$ and the *intersection* $\bigcap s_i$ of a family $(s_i)_{i \in I}$ of fuzzy subsets by

$$\bigcup s_i(x) = \sup\{s_i(x) | i \in I\} \quad \text{and} \quad \bigcap s_i(x) = \inf\{s_i(x) | i \in I\}.$$

Also, the *complement* $\sim s$ of a fuzzy subset s is defined by setting

$$(\sim s)(x) = 1 - s(x)$$

for every $x \in S$. We say that s is *finite* if its support $\mathrm{Supp}(s) = \{x \in S \mid s(x) \neq 0\}$ is finite, and that s is a *fuzzy singleton* if Supp(s) contains only one element. Namely, if $c \in S$ and $\lambda \in U$, $\lambda \neq 0$, then $\{c\}^\lambda : S \to U$ denotes the singleton defined by $\{c\}^\lambda(c) = 1$ and $\{c\}^\lambda(x) = 0$ if $x \neq c$.

Following Pavelka [8] (see also the book of L. Bolc and P. Borowik [4]), we define the notion of fuzzy semantics in a very abstract way. The idea is that a semantics is defined by a suitable set of possible models for the language under consideration and that, in turn, every model can be identified with the valuation of the formulas in U it determines (i.e. a fuzzy subset of formulas). So, if $\mathbb{F}$ is a set whose elements we call *formulas*, then a *fuzzy semantics* is any class $\mathcal{M}$ of fuzzy subsets of $\mathbb{F}$. We use the name *models* or *worlds* for the elements of $\mathcal{M}$, and given a formula α and a world m, the number $m(\alpha)$ represents the *truth degree* of α in m. In this way fuzzy logic looks similar to multivalued logic, since it considers the possibility that formulas are valued in a structure different from the Boolean algebra $\{0, 1\}$. As a matter of fact, however, the point of view of fuzzy logic is quite different. Indeed, the basic idea (in my opinion) is that the available information is a constraint $\mathfrak{C}$ on the possible truth values of the formulas in the actual world and that a deductive apparatus is a tool to obtain an improved constraint $\overline{\mathfrak{C}}$. In logical terms, while the constraint $\mathfrak{C}$ corresponds to the system of axioms, the derived constraint $\overline{\mathfrak{C}}$ corresponds to the set of logical consequences of $\mathfrak{C}$. Namely, fuzzy logic considers only constraints like

"the actual truth value of the formula α is at least $v(\alpha)$"

where v is a fuzzy subset of formulas that one calls a *fuzzy system of proper axioms*, or *initial valuation*. On the other hand, also in the classical case a system Σ of axioms is a constraint like "the actual truth value of the formula α is at least $\chi_\Sigma(\alpha)$." So, in accordance with this interpretation, we say that $m \in \mathcal{M}$ is a *model* of v, and we write $m \vdash v$ provided that $v \subseteq m$. In this case v represents correct (partial) fuzzy information about the world m. We say that v is *satisfiable* if it has a model in $\mathcal{M}$, and we denote by $Sat(\mathcal{M})$ the class of satisfiable initial valuations. Given a fuzzy semantics $\mathcal{M}$, we define a *theory* as any fuzzy subset of formulas obtained as an intersection of elements of $\mathcal{M}$. In particular, since the intersection of the empty subset of $\mathcal{M}$ gives the fuzzy subset constantly equal to 1 (that is, the whole set of formulas $\mathbb{F}$), such a fuzzy subset is a theory, which we call *the inconsistent theory*. Also, all the elements of $\mathcal{M}$ are theories, obviously. Given a fuzzy semantics, a *logical consequence operator* $C : \mathcal{F}(\mathbb{F}) \to \mathcal{F}(\mathbb{F})$ is defined by setting, for every $v \in \mathcal{F}(\mathbb{F})$,

$$C(v) = \bigcap \{m \in \mathcal{M} | m \vdash v\}. \tag{2.2}$$

$C(v)$ is a theory, which we call the *fuzzy theory generated by* v. In particular, if v is not satisfiable, then $C(v)$ collapses in the inconsistent theory.

Notice that the above definitions are natural extensions of the corresponding ones in classical logic. Indeed, in classical logic every model m defines the complete theory $\{\alpha \in \mathbb{F} \mid \alpha$ is true in $m\}$ of m, and every complete theory can be obtained in this way by a suitable model. So the semantics of a first order logic coincides with the class of the complete theories, in a sense. Also, given a system Σ of axioms (the available information), a model of Σ is a complete theory containing Σ. Finally, the theory $C(\Sigma)$ generated by Σ, i.e. the set of logical consequences of Σ, can be defined as the intersection of all the models (i.e. the complete theories) containing Σ.

As in the classical case, we say that $\mathcal{M}$ is *compact* if given any initial valuation v, v is satisfiable if and only if every finite fuzzy subset of v is satisfiable. A more interesting concept of compactness is defined as follows. Recall that a family $(v_i)_{i \in I}$ of fuzzy subsets is called *directed* if for every $i, j \in I$ an element $h \in I$ exists such that $v_i \subseteq v_h$ and $v_j \subseteq v_h$. Moreover, we define the *limit* of a directed family $(s_i)_{i \in I}$ as the union $\bigcup\{s_i \mid i \in I\}$. Finally, a class $\mathcal{K}$ of fuzzy subsets is called *inductive* if the limit of an inductive family of elements in $\mathcal{K}$ is an element in $\mathcal{K}$. We say that a fuzzy semantics $\mathcal{M}$ is *strongly compact*, in brief *s-compact*, if $Sat(\mathcal{M})$ is inductive. The following proposition gives a simple characterization of the s-compact semantics where, for every s_1, s_2 in $\mathcal{F}(\mathbb{F})$, we set $s_1 \ll s_2$ provided that $s_1(x) < s_2(x)$ for every $x \in Supp(s_1)$.

PROPOSITION 2.1 *A fuzzy semantics is s-compact if and only if*

$$v \text{ is satisfiable} \quad \Leftrightarrow \quad \text{every finite } v_f \ll v \text{ is satisfiable}. \tag{2.3}$$

In particular, every s-compact fuzzy semantics is compact (while the converse implication falls).

Proof Let $\mathcal{M}$ be s-compact, and assume that every finite fuzzy subset v_f such that $v_f \ll v$ is satisfiable. Then, since v is the inductive limit of the directed class $\{v_f \mid v_f \ll v,\ v_f$ finite$\}$, we have $v \in Sat(\mathcal{M})$. This proves (2.3).

Conversely, assume (2.3), and let $\mathcal{H}$ be an inductive class of satisfiable fuzzy subsets. We have to prove that $v = \bigcup \mathcal{H}$ is satisfible. Now, for every finite fuzzy set v_f such that $v_f \ll v$, an element $f \in \mathcal{H}$ exists such that $v_f \subseteq f$. Since f is satisfiable, v_f is satisfiable too, and by (2.3) v is satisfiable.

By setting $\mathcal{M} = \{s \in \mathcal{F}(\mathbb{F}) \mid s(\alpha) \neq 1$ for every $\alpha \in \mathbb{F}\}$ we obtain an example of compact fuzzy semantics that is not s-compact. ∎

An interesting characteristic of an s-compact semantics is that the associated logical consequence operator C is continuous, that is, the limits of the directed families of initial valuations are preserved. In particular, since the class of all the finite fuzzy subsets of a given fuzzy subset is directed, we have that

$$C(v) = \bigcup \left\{C(v_f) | v_f \text{ is a finite fuzzy subset of } v\right\}.$$

The next proposition gives a simple criterion to prove the s-compactness of a fuzzy semantics (see also [1]). Recall that a class $\mathcal{F}$ of subsets of N is an *ultrafilter* provided that

(1) $X \in \mathcal{F}, Y \supseteq X \ \Rightarrow\ Y \in \mathcal{F}$;

(2) $X \in \mathcal{F}, Y \in \mathcal{F} \ \Rightarrow\ X \cap Y \in \mathcal{F}$;

(3) for every X, either $X \in \mathcal{F}$ or $-X \in \mathcal{F}$.

Moreover, given a sequence $(\lambda_n)_{n \in N}$ of elements of U, we write $\lim_{\mathcal{F}} \lambda_n = \lambda$ provided that

$$\forall \epsilon > 0 \quad \exists X \in \mathcal{F} \quad \forall n \in X \qquad |\lambda - \lambda_n| \leq \epsilon.$$

This notion of convergence satisfies the same properties of the classical one, but, in addition, $\lim_{\mathcal{F}} \lambda_n$ exists for any sequence $(\lambda_n)_{n \in N}$ (for example, see Chang and Keisler [5]). Given a sequence $(s_n)_{n \in N}$ of fuzzy subsets of $\mathbb{F}$ and a prime filter $\mathcal{U}$, we define the *ultraproduct modulo* of $\mathcal{U}$ of $(s_n)_{n \in N}$ as the fuzzy subset

$$s(x) = \lim_{\mathcal{U}} s_n(x), \qquad x \in \mathbb{F}.$$

PROPOSITION 2.2 *Let $\mathcal{M}$ be a fuzzy semantics closed with respect to the ultraproducts. Then $\mathcal{M}$ is s-compact and therefore compact. Moreover, for every formula α and an initial valuation v, a model m of v exists such that $m(\alpha) = C(v)(\alpha)$.*

Proof Let v be an initial valuation such that every finite fuzzy subset v_f such that $v_f \ll v$ is satisfiable, and denote by $\alpha_1, \alpha_2, \ldots$ an enumeration of all the formulas in $\mathbb{F}$. Moreover, for every $h \in N$, consider the fuzzy subset v_h defined by

$$v_h(\alpha) = \begin{cases} 0 \vee \left(v(\alpha_i) - \dfrac{1}{h} \right) & \text{if} \quad \alpha = \alpha_i \text{ with } i \le h, \\ 0 & \text{otherwise.} \end{cases}$$

Then, since v_h is finite and $v_h \ll v$, an element $m_h \in \mathscr{M}$ exists such that

$$m_h(\alpha_1) \ge v(\alpha_1) - \frac{1}{h}, \dots, \qquad m_h(\alpha_h) \ge v(\alpha_h) - \frac{1}{h}.$$

Let $\mathscr{U}$ be a nonprincipal prime filter, and let m the ultraproduct of the sequence $(m_n)_{n \in N}$ modulo $\mathscr{U}$. In order to prove that m is a model of v, notice that, given any formula α_j, we have $m_h(\alpha_j) \ge v(\alpha_j) - 1/h$ for every $h \ge j$, and therefore $m(\alpha_j) = \lim_{\mathscr{U}} m_n(\alpha_j) \ge v(\alpha_j)$.

Let v be an initial valuation and α a formula. Then, since $C(v)(\alpha) = \inf\{m(\alpha) \mid m \in \mathscr{M}, m \supseteq v\}$, a sequence m_n of models of v exists such that $m_n(\alpha)$ is a decreasing sequence of numbers such that $\lim_{n \to \infty} m_n(\alpha) = C(v)(\alpha)$. Let $\mathscr{U}$ be a nonprincipal ultrafilter, and m the ultrapower of $(m_n)_{n \in N}$ modulo $\mathscr{U}$. Then

$$m(\alpha) = \lim_{\mathscr{U}} m_n(\alpha) = \lim_{n \to \infty} m_n(\alpha) = C(v)(\alpha). \qquad \blacksquare$$

Notice that the class of fuzzy semantics that are closed with respect to the ultraproduct is very large. As a matter of fact if $\mathscr{M}$ is defined by a set of equalities or inequalities and some continuity hypothesis is satisfied, then $\mathscr{M}$ is closed with respect to the ultraproducts.

In the sequel we assume that among the logical connectives there is a *negation* $\sim$, that is, for every formula α in $\mathbb{F}$, $\sim \alpha$ is a formula in $\mathbb{F}$. Given a fuzzy subset s of formulas, we denote by s^* the fuzzy subset defined by

$$s^*(\alpha) = 1 - s(\sim \alpha), \qquad \alpha \in \mathbb{F}.$$

Then we say that the formula α is *decidable* in v if $C(v)(\alpha) = C(v)^*(\alpha)$, that is, if $C(v)(\alpha) + C(v)(\sim \alpha) = 1$; that v is *complete* (with respect to the negation) if every formula is decidable in v; and that $\mathscr{M}$ is *balanced* if every element of $\mathscr{M}$ is complete. We define an *interval constraint* as any pair (l, u) of fuzzy subsets of formulas, and we say that $m \in \mathscr{M}$ *satisfies* (l, u) provided that $l(\alpha) \le m(\alpha) \le u(\alpha)$ for every $\alpha \in \mathbb{F}$. The following proposition shows that for balanced fuzzy semantics the initial valuations are equivalent to the interval constraints.

PROPOSITION 2.3 *Assume that $\mathcal{M}$ is a balanced fuzzy semantics, and let v be an initial valuation. Then, for $m \in \mathcal{M}$,*

$$m \text{ is a model of } v \quad \Leftrightarrow \quad m \text{ satisifies } (v, v^*).$$

Moreover, if (l, u) is an interval constraint, then

$$m \text{ satisfies } (l, u) \quad \Leftrightarrow \quad m \text{ is a model of } l \cup u^*.$$

Proof Let v be an initial valuation. Then, since from $m \supseteq v$ it follows that $1 - m(\alpha) = m(\sim \alpha) \geq v(\sim \alpha)$ and therefore $m(\alpha) \leq v^*(\alpha)$, we have that m is a model of v if and only if $m(\alpha) \in [v(\alpha), v^*(\alpha)]$ for every formula α. Besides, assume that (l, u) is an interval constraint and that m satisfies (l, u). Then, since $m(\alpha) = 1 - m(\sim \alpha) \geq 1 - u(\sim \alpha) = u^*(\alpha)$, m is a model of $l \cup u^*$. Conversely, if m is a model of $1 \cup u^*$, then, since $m(\alpha) = 1 - m(\sim \alpha) \leq 1\text{-}u^*(\sim \alpha) = u(\alpha)$, m satisfies (l, u). ■

In accordance with Proposition 2.3, an initial valuation v for an unknown world m gives for every formula α a constraint $[v(\alpha), v^*(\alpha)]$ for $m(\alpha)$. Since m is also a model of $C(v)$, another constraint is given by $[C(v)(\alpha), C(v)^*(\alpha)]$. Now, $[C(v)(\alpha), C(v)^*(\alpha)]$ is contained in $[v(\alpha), v^*(\alpha)]$, and this means that the operator C is a tool to improve the initial interval approximations of the unknown truth values. Namely, it gives the best possible approximation of m given the information v. If α is decidable in v, the interval $[C(v)(\alpha), C(v)^*(\alpha)]$ becomes a number and $m(\alpha)$ is completely determined. If v is complete, then there is only one model of v and this model is $C(v)$. Obviously, it is also possible that v is not satisfiable, that is, no model m exists such that $m(\alpha) \in [v(\alpha), v^*(\alpha)]$ for every formula α. This happens, for example, if $[v(\alpha), v^*(\alpha)]$ is empty, that is, $v(\alpha) + v(\sim \alpha) > 1$.

PROPOSITION 2.4 *If $\mathcal{M}$ is balanced and closed with respect to the ultraproducts, then for every initial valuation v and $\alpha \in \mathbb{F}$ two models m_1 and m_2 of v exist such that*

$$C(v)(\alpha) = m_1(\alpha) \quad \text{and} \quad C(v)^*(\alpha) = m_2(\alpha).$$

Proof The first equality is a consequence of Proposition 2.2. Let m_2 be a model of v such that $m_2(\sim \alpha) = C(v)(\sim \alpha)$. Then

$$m_2(\alpha) = 1 - m_2(\sim \alpha) = C(v)^*(\alpha). \qquad ■$$

3. REFUTATION PROCEDURES

A *fuzzy logic* is usually defined by adding to a fuzzy semantics a *fuzzy syntax*, that is, a suitable fuzzy subset of logical axioms and suitable fuzzy inference rules. In this section we will examine the possibility of replacing this "Hilbert style" with a "refutation style" approach. To this aim, given a fixed, fuzzy semantics $\mathcal{M}$, we have to introduce the basic notion of degree of consistency of a formula with respect to an initial valuation.

DEFINITION 3.1 *Let v be an initial valuation and α a formula. Then the degree of consistency of α with v is the number*

$$Cons(v, \alpha) = \sup\{\lambda \in U | v \cup \{\alpha\}^{\lambda} \textit{ is satisfiable}\}, \tag{3.1}$$

while the degree of inconsistency of α with v is $Inc(v, \alpha) = 1 - Cons(v, \alpha)$, that is,

$$Inc(v, \alpha) = \inf\{\lambda \in U | v \cup \{\alpha\}^{1-\lambda} \textit{ is satisfiable}\}. \tag{3.2}$$

Obviously, we have that

$$\begin{aligned} v \text{ is not satisfiable} \quad &\Leftrightarrow \quad \mathrm{Inc}(v, \alpha) = 1 \quad \forall \alpha \in \mathbb{F} \\ &\Leftrightarrow \quad \mathrm{Cons}(v, \alpha) = 0 \quad \forall \alpha \in \mathbb{F}. \end{aligned}$$

PROPOSITION 3.2 *For every initial valuation v and $\alpha \in \mathbb{F}$,*

$$Cons(v, \alpha) = \inf\{\lambda \in U | v \cup \{\alpha\}^{\lambda} \textit{ is not satisfiable}\}, \tag{3.3}$$

$$Inc(v, \alpha) = \sup\{\lambda \in U | v \cup \{\alpha\}^{1-\lambda} \textit{ is not satisfiable}\}. \tag{3.4}$$

Proof The set $\{\lambda \in U \mid v \cup \{\alpha\}^{\lambda}$ is satisfiable$\}$ is an interval; indeed, if $\mu < \lambda$ and $v \cup \{\alpha\}^{\lambda}$ is satisfiable, then $v \cup \{\alpha\}^{\mu}$ is satisfiable. Consequently

$$\begin{aligned} &\sup\{\lambda \in U | v \cup \{\alpha\}^{\lambda} \text{ is satisfiable}\} \\ &\qquad = \inf\{\lambda \in U | v \cup \{\alpha\}^{\lambda} \text{ is not satisfiable}\}. \end{aligned}$$

Likewise one proceeds to prove (3.4). ∎

The following proposition shows that, in a sense, any balanced semantics admits a refutation procedure.

PROPOSITION 3.3 *Let $\mathcal{M}$ be a balanced semantics. Then*

$$C(v)(\alpha) = \mathrm{Inc}(v, \sim \alpha), \tag{3.5}$$

and therefore

$$C(v)^*(\alpha) = \mathrm{Cons}(v, \alpha). \tag{3.6}$$

Proof Assume that $\lambda < C(v)(\alpha)$. Then for every model m of v we have that $m(\alpha) > \lambda$ and therefore $m(\sim \alpha) = 1 - m(\alpha) < 1 - \lambda$. This means that $v \cup \{\sim \alpha\}^{1-\lambda}$ is not satisfiable and therefore that

$$C(v)(\alpha) = \sup\{\lambda \in U \mid \lambda < C(v)(\alpha)\}$$
$$\leq \sup\{\lambda \in U \mid v \cup \{\sim \alpha\}^{1-\lambda} \text{ is not satisfiable}\} = \mathrm{Inc}(v, \sim \alpha).$$

Assume that $v \cup \{\sim \alpha\}^{1-\lambda}$ is not satisfiable. Then for every model m of v, we have $m(\sim \alpha) < 1 - \lambda$ and therefore $m(\alpha) = 1 - m(\sim \alpha) > \lambda$. Thus $C(v)(\alpha) \geq \lambda$, and this proves that $C(v)(\alpha) \geq \mathrm{Inc}(v, \sim \alpha)$.

(3.6) is an immediate consequence of (3.5). ■

From (3.5) and (3.6) it follows that if v is a fuzzy set representing the available information about an unkown world m, then the best constraint we may obtain to the actual truth degree $m(\alpha)$ is furnished by the degree of inconsistency of $\sim \alpha$ and the degree of consistency of α. Obviously, the basic question is to find effective methods to compute the functions Inc and Cons.

4. PROBABILITY LOGIC: THE SEMANTICS

In this section we assume that $\mathbb{F}$ is the set of formulas of a zero order calculus whose propositional variables are $p_1, p_2, \ldots$ and whose connectives are $\wedge$, $\vee$, and $\sim$. We write $a \underset{\equiv}{<} \beta$ to say that α implies β, and $\alpha \equiv \beta$ to say that α is logically equivalent to β. Also, we denote by **0** and **1** a fixed contradiction and tautology, respectively. In accordance with the literature (see for example [6]), the notion of probability model is on the basis of probability logic. A *probability model* is a fuzzy subset p: $\mathbb{F} \to U$ such that

(i) $p(\mathbf{1}) = 1$,

(ii) $\alpha \equiv \beta \Rightarrow p(\alpha) = p(\beta)$ (transparency),

(iii) α and β incompatible $\Rightarrow p(\alpha \vee \beta) = p(\alpha) + p(\beta)$ (finite additivity).

We consider the fuzzy semantics defined by the set $\mathbb{P}$ of probability models. In accordance with the fuzzy logic point of view, such a semantics gives the logical consequence operator C defined by

$$C(v) = \bigcap \{p \mid p \in \mathbb{P}, p \vdash v\}.$$

We call the theory $C(v)$ the *probability theory generated by* v. It is immediate that $C(v)(\mathbf{1}) = 1$ and that $\alpha \equiv \beta$ entails $C(v)(\alpha) = C(v)(\beta)$, but, since it is impossible that $C(v)(\alpha) + C(v)(\sim \alpha) \neq 1$ and even $C(v)(\alpha) = C(v)(\sim \alpha) = 0$, the map $C(v)$ is not finitely additive in general. The following proposition is immediate.

PROPOSITION 4.1 *The class $\mathbb{P}$ of probability models is a balanced fuzzy semantics closed with respect to the ultraproducts. As a consequence, an initial valuation v is satisfiable if and only if every finite part of v is satisfiable. Moreover, given an initial valuation v, for every formula α two probability models p and p' exist such that $C(v)(\alpha) = p(\alpha)$ and $C(v)^*(\alpha) = p'(\alpha)$.*

REMARK The probability models coincide with the finitely additive probabilities in a Boolean algebra, in a sense. Indeed, denote by **B** the Lindenbaum algebra of the classical propositional calculus, that is, the quotient of the free algebra $(\mathbb{F}, \wedge, \vee, \sim)$ modulo $\equiv$. Namely, $\mathbf{B} = \{[\alpha] \mid \alpha \in \mathbb{F}\}$, where $[\alpha] = \{\beta \in \mathbb{F} \mid \beta \equiv \alpha\}$; the Boolean operations are defined by $[\alpha] \wedge [\beta] = [\alpha \wedge \beta]$, $[\alpha] \vee [\beta] = [\alpha \vee \beta]$, $\sim [\alpha] = [\sim \alpha]$; and the minimum $\mathbf{0}$ and the maximum $\mathbf{1}$ are the classes $[\mathbf{0}]$ and $[\mathbf{1}]$, respectively. It is immediate that if $p \in \mathbb{P}$, then by setting $\mu([\alpha]) = p(\alpha)$ we obtain a finitely additive probability. Conversely, if $\mu : B \to [0,1]$ is a finitely additive probability, then the map p defined by setting $p(\alpha) = \mu([\alpha])$ is a probability model.

Likewise one proves that the probability theories coincide with the lower envelopes (that is, the maps that can be obtained as a least upper bound of a family of probabilities). In [7] probability logic is examined, in a Hilbert style, by assuming directly that the set of formulas is a Boolean algebra.

The following definitions will be useful in the sequel. For every $h \in N$ and $k \in N_o$ define the *h-k-connective* as the *h-ary* operation $C^k : \mathbb{F}^h \to \mathbb{F}$ specified by setting

$$C^0(\alpha_1, \dots, \alpha_h) = \mathbf{1},$$

$$C^{h+1}(\alpha_1, \dots, \alpha_h) = C^{h+2}(\alpha_1, \dots, \alpha_h) = \cdots = \mathbf{0},$$

and, for $k = 1, \dots, h$,

$$C^k(\alpha_1, \dots, \alpha_h) = \bigvee \left\{\alpha_{i_1} \wedge \cdots \wedge \alpha_{i_k} | i_1, \dots, i_k \text{ are distinct}\right\}.$$

In other words,

$$C^1(\alpha_1, \dots, \alpha_h) = \alpha_1 \vee \cdots \vee \alpha_h,$$

$$C^2(\alpha_1, \dots, \alpha_h) = \bigvee \left\{\alpha_i \wedge \alpha_j | i \neq j, i, j \in \{1, \dots, h\}\right\},$$

$$\vdots$$

$$C^h(\alpha_1, \dots, \alpha_h) = \alpha_1 \wedge \cdots \wedge \alpha_h.$$

It is immediate to prove that

$$C^0(\alpha_1,\ldots,\alpha_h) \geqq C^1(\alpha_1,\ldots,\alpha_h) \geqq \cdots \geqq C^h(\alpha_1,\ldots,\alpha_h),$$

$$C^k(\alpha_1,\ldots,\alpha_h) \geqq C^k(\alpha_1,\ldots,\alpha_{h-1}),$$

$$C^k(\alpha_1,\ldots,\alpha_h) \equiv C^k(\alpha_1,\ldots,\alpha_h,0) \equiv C^{k+1}(\alpha_1,\ldots,\alpha_h,1).$$

The connection between such connectives and the probability models is expressed by the following proposition.

PROPOSITION 4.2 *Let $p : \mathbb{F} \to U$ be a map satisfying* (i) *and* (ii) *such that $p(\mathbf{0}) = 0$. Then p is a probability model if and only if, for every $\alpha_1,\ldots,\alpha_h$ in $\mathbb{F}$,*

$$p(\alpha_1) + \cdots + p(\alpha_h) = p(C^1(\alpha_1,\ldots,\alpha_h)) + \cdots + p(C^h(\alpha_1,\ldots,\alpha_h)). \tag{4.1}$$

Proof If p satisfies (4.1), then for $h = 2$,

$$p(\alpha_1) + p(\alpha_2) = p(\alpha_1 \vee \alpha_2) + p(\alpha_1 \wedge \alpha_2). \tag{4.2}$$

Consequently, if α_1 is inconsistent with α_2,

$$p(\alpha_1) + p(\alpha_2) = p(\alpha_1 \vee \alpha_2) + p(\mathbf{0}) = p(\alpha \vee \alpha_2)$$

and p is a probability model.

Conversely, let p be a probability model. Then if $h = 1$, the condition (4.1) is immediate. Moreover,

$$p(\alpha_1) = p(\alpha_1 \wedge \sim \alpha_2) + p(\alpha_1 \wedge \alpha_2),$$

$$p(\alpha_2) = p(\alpha_2 \wedge \sim \alpha_1) + p(\alpha_2 \wedge \alpha_1),$$

$$p(\alpha_1 \vee \alpha_2) = p(\alpha_1 \wedge \sim \alpha_2) + p(\alpha_2 \wedge \sim \alpha_1) + p(\alpha_1 \wedge \alpha_2),$$

and therefore

$$\begin{aligned} p(\alpha_1 \vee \alpha_2) &= p(\alpha_1) - p(\alpha_1 \wedge \alpha_2) + p(\alpha_2) - p(\alpha_2 \wedge \alpha_1) + p(\alpha_1 \wedge \alpha_2) \\ &= p(\alpha_1) + p(\alpha_2) - p(\alpha_1 \wedge \alpha_2). \end{aligned}$$

This proves (4.2), that is, (4.1) in case $h = 2$. To proceed by induction on h, observe that

$$C^i(\alpha_1,\ldots,\alpha_h) \equiv \left[\alpha_h \wedge C^{i-1}(\alpha_1,\ldots,\alpha_{h-1})\right] \vee C^i(\alpha_1,\ldots,\alpha_{h-1}),$$

and therefore, since $\alpha_h \wedge C^{i-1}(\alpha_1,\dots,\alpha_{h-1}) \wedge C^i(\alpha_1,\dots,\alpha_{h-1})$ is logically equivalent to $\alpha_h \wedge C^i(\alpha_1,\dots,\alpha_{h-1})$, we have

$$p(C^i(\alpha_1,\dots,\alpha_h)) = p(\alpha_h \wedge C^{i-1}(\alpha_1,\dots,\alpha_{h-1})) + p(C^i(\alpha_1,\dots,\alpha_{h-1})) - p(\alpha_h \wedge C^i(\alpha_1,\dots,\alpha_{h-1})).$$

Then

$$\begin{aligned}\sum_{i=1}^{i=h} p(C^i(\alpha_1,\dots,\alpha_h)) &= \sum_{i=1}^{i=h} p(\alpha_h \wedge C^{i-1}(\alpha_1,\dots,\alpha_{h-1})) \\ &\quad + \sum_{i=1}^{i=h} p(C^i(\alpha_1,\dots,\alpha_{h-1})) \\ &\quad - \sum_{i=1}^{i=h} p(\alpha_h \wedge C^i(\alpha_1,\dots,\alpha_{h-1})) \\ &= p(\alpha_h) + \sum_{i=1}^{i=h} p(C^i(\alpha_1,\dots,\alpha_{h-1})) \\ &= p(\alpha_h) + \sum_{i=1}^{i=h-1} p(C^i(\alpha_1,\dots,\alpha_{h-1})).\end{aligned}$$

Since, by the inductive hypothesis, $\sum_{i=1}^{i=h-1} p(C^i(\alpha_1,\dots,\alpha_{h-1})) = \sum_{i=1}^{i=h-1} p(\alpha_i)$, Equation (4.1) is proved. ∎

Now, we will define a useful function M enabling us to give a simple characterization of the satisfiable fuzzy subsets of formulas. Similar characterizations can be found in [1] and [7].

Given the formulas $\alpha_1,\dots,\alpha_h$, we set

$$M(\alpha_1,\dots,\alpha_h) = \max\{k \in N \mid C^k(\alpha_1,\dots,\alpha_h) \text{ is consistent}\}, \quad (4.3)$$

or, equivalently,

$$M(\alpha_1,\dots,\alpha_h) = \min\{k \in N \mid C^k(\alpha_1,\dots,\alpha_h) \text{ is a contradiction}\} - 1. \quad (4.4)$$

Such a map is compatible with $\equiv$, that is,

$$\alpha_1 \equiv \beta_1,\dots,\alpha_h \equiv \beta_h \quad \Rightarrow \quad M(\alpha_1,\dots,\alpha_h) = M(\beta_1,\dots,\beta_h).$$

It is also possible to define $M(\alpha_1,\dots,\alpha_h)$ by induction on h. Namely, in case $h = 1$ we have

$$M(\alpha_1) = \begin{cases} 0 & \text{if } \alpha_1 \text{ is a contradiction,} \\ 1 & \text{otherwise,} \end{cases}$$

while for $h \neq 1$

$$M(\alpha_1, \dots, \alpha_{h+1}) = \begin{cases} M(\alpha_1, \dots, \alpha_h) & \text{if } \alpha_{h+1} \text{ contradicts} \\ & \quad C^{M(\alpha_1, \dots, \alpha_h)}(\alpha_1, \dots, \alpha_h), \\ M(\alpha_1, \dots, \alpha_h) + 1 & \text{otherwise.} \end{cases}$$

PROPOSITION 4.3 *A fuzzy set v of formulas is satisfiable if and only if, for every $\alpha_1, \dots, \alpha_h$ in Supp(v),*

$$v(\alpha_1) + \cdots + v(\alpha_h) \le M(\alpha_1, \dots, \alpha_h). \tag{4.5}$$

Proof Assume v satisfiable, and let p be a probability model of v. Then, if we set $m = M(\alpha_1, \dots, \alpha_h)$, since $C^{m+1}(\alpha_1, \dots, \alpha_h), \dots, C^h(\alpha_1, \dots, \alpha_h)$ are contradictions, we have

$$\begin{aligned} v(\alpha_1) + \cdots + v(\alpha_h) &\le p(\alpha_1) + \cdots + p(\alpha_h) \\ &\le p(C^1(\alpha_1, \dots, \alpha_h)) + \cdots + p(C^m(\alpha_1, \dots, \alpha_h)) \le m. \end{aligned}$$

To prove the converse part of the proposition it is enough to prove that, for every finite set of formulas $\beta_1, \dots, \beta_t$ in Supp(v), a probability model p exists such that

$$p(\beta_1) \ge v(\beta_1), \dots, p(\beta_t) \ge v(\beta_t). \tag{4.6}$$

Let k be the least integer such that the propositional variables occurring in $\beta_1, \dots, \beta_t$ are in $\{p_1, \dots, p_k\}$, and set

$$E = \left\{ p_1^{i_1} \cdots \wedge p_k^{i_k} \middle| (i_1, \dots, i_k) \in \{-1, 1\}^k \right\},$$

where, for every formula α, we set $\alpha^1 = \alpha$ and $\alpha^{-1} = \sim \alpha$. If we denote by $e_1, \dots, e_n$ the elements of E, it is immediate that any two elements of E are incompatible and that $e_1 \vee \cdots \vee e_n$ is a tautology. Moreover, the disjunctive normal form theorem claims that every formula β_j is equivalent to a disjunction of formulas in E namely, the disjunction of the formulas e_i such that $e_i \underset{\equiv}{<} \beta_j$. Now, by setting $a_i^j = 1$ if $e_i \underset{\equiv}{<} \beta_j$ and $a_i^j = 0$ otherwise, we have that a probability model p satisfies (4.6) if and only if

$$\begin{gathered} a_1^1 p(e_1) + \cdots + a_n^1 p(e_n) \ge v(\beta_1), \\ \vdots \\ a_1^t p(e_1) + \cdots + a_n^t p(e_n) \ge v(\beta_t). \end{gathered}$$

This suggests searching for a solution of the following system:

$$\begin{aligned} a_1^1 x_1 + \cdots + a_n^1 x_n &\geq v(\beta_1), \\ &\vdots \\ a_1^t x_1 + \cdots + a_n^t x_n &\geq v(\beta_t) \end{aligned} \tag{4.7}$$

with the constraints $x_1 \geq 0, \ldots, x_n \geq 0$, and $x_1 + \cdots + x_n = 1$. Now, we claim that (4.5) entails that such a solution exists. Indeed, by setting $b_i^j = a_i^j - v(\beta_j)$, since $x_1 + \cdots + x_n = 1$ and therefore $v(\beta_j) = x_1 v(\beta_j) + \cdots + x_n v(\beta_j)$, Equation (4.7) becomes

$$\begin{aligned} b_1^1 x_1 + \cdots + b_n^1 x_n &\geq 0, \\ &\vdots \\ b_1^t x_1 + \cdots + b_n^t x_n &\geq 0. \end{aligned} \tag{4.8}$$

Now, recall that the fundamental theorem of linear inequalities assures that if a vector z cannot be obtained as a nonnegative linear combination of the vectors $x_1, \ldots, x_n$, then a vector c exists such that $c \cdot z < 0$, $c \cdot \mathrm{x}_1 \geq 0, \ldots, c \cdot x_n \geq 0$ (see for example [11]). Denote by b^j the vector $(b_1^j, \ldots, b_n^j)$ for $j = 1, \ldots, t$, and by u^j the unitary vector $(\delta_{1j}, \ldots, \delta_{nj})$, where, as usual, $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ otherwise; and let z be equal to $(-1, \ldots, -1) \in R^n$. We claim that z cannot be obtained as a nonnegative linear combination of $b^1, \ldots, b^t, u^1, \ldots, u^n$ and therefore that a vector $c = (c_1, \ldots, c_n)$ exists such that

$$\sum_i c_i > 0, \qquad c_1 \geq 0, \ldots, \quad c_n \geq 0, \qquad \sum_i b_i^1 c_i \geq 0, \ldots, \qquad \sum_i b_i^t c_i \geq 0.$$

Indeed, otherwise nonnegative numbers $y_1, \ldots, y_t$ and a nonnegative vector y should exist such that $y_1 b^1 + \cdots + y_t b^t + y = z$. Then the set

$$S = \{(y_1, \ldots, y_t) | y_i \geq 0, \, y_1 b^1 + \cdots + y_t b^t < 0\}$$

should be nonempty and, since S is open on $[0, \infty)^t$, in S there are vectors whose components are rational numbers. Also, since

$$(y_1, \ldots, y_t) \in S, \quad \lambda > 0 \quad \Rightarrow \quad (\lambda y_1, \ldots, \lambda y_t) \in S,$$

in S there is a vector whose components are natural numbers $\lambda_1, \ldots, \lambda_t$. Consider the sequence $\alpha_1, \ldots, \alpha_h$ obtained by considering λ_1 formulas equal to β_1, λ_2 formulas equal to β_2, and so on, and set $m = M(\alpha_1, \ldots, \alpha_h)$. Then there is a conjunction $\alpha_{i_1} \wedge \cdots \wedge \alpha_{i_m}$ of m formulas in $\alpha_1, \ldots, \alpha_h$ that is a consistent formula. Let e_i be an element of E

occurring in the disjunctive normal form of $\alpha_{i_1} \wedge \cdots \wedge a_{i_m}$. Then, since e_i implies exactly m formulas in $\alpha_1, \ldots, \alpha_h$, we have that $m = a_i^1\lambda_1 + \cdots + a_i^t\lambda_t$, and therefore, by (4.5),

$$a_i^1\lambda_1 + \cdots + \alpha_i^t\lambda_t - [\lambda_1 v(\beta_1) + \cdots + \lambda_t v(\beta_t)]$$
$$= \lambda_1\left[a_i^1 - v(\beta_1)\right] + \cdots + \lambda_t\left[a_i^t - v(\beta_t)\right]$$

is nonnegative. Since such a number coincides with the i-component of $\lambda_1 b^1 + \cdots + \lambda_t b^t$, this is an absurdity.

By setting $m_i = c_i/(c_1 + \cdots + c_n)$ we obtain a solution of (4.7) satisfying the required constraints.

Turning back to the question of finding a probability model satisfying (4.6), given $e_i = p_1^{i_1} \wedge \cdots \wedge p_k^{i_k}$, we denote by v_i the 0-1-valuation of the formulas defined by setting

$$v_i(p_j) = \begin{cases} i_j & \text{if } j \le k, \\ 0 & \text{otherwise}. \end{cases}$$

It is immediate that $v_i(e_i) = 1$ and $v_i(e_j) = 0$ for $j \neq i$. Moreover, define p by

$$p(\alpha) = \begin{cases} 0 & \text{if } \alpha \text{ is a contradiction} \\ \sum\{m_i | v_i(\alpha) = 1\} & \text{otherwise}. \end{cases}$$

Since $m_1 + \cdots + m_n = 1$, we have that $p(\alpha) = 1$ for every tautology α. Moreover, it is immediate that $\alpha \equiv \beta$ implies $p(\alpha) = p(\beta)$. Let α, β be incompatible formulas; then since no 0-1-valuation satisfies both α and β,

$$p(\alpha \vee \beta) = \sum\{m_i | v_i(\alpha \vee \beta) = 1\}$$
$$= \sum\{m_i | v_i(\alpha) = 1\} + \sum\{m_i | v_i(\beta) = 1\} = p(\alpha) + p(\beta),$$

and this proves that p is a probability model. Also,

$$p(\beta_j) = \sum\left\{m_i | v_i(\beta_j) = 1\right\}$$
$$= \sum\left\{m_i | e_i \underset{\equiv}{<} \beta_j\right\} = a_1^j m_1 + \cdots + a_n^j m_n \ge v(\beta_j),$$

and this concludes the proof. ∎

Observe that if (4.5) is satisfied by all the formulas in Supp(v), then (4.5) is satisfied for all the formulas. Indeed, assume that $\alpha_1, \ldots, \alpha_h$ satisfy (4.5) and that $v(\alpha_{h+1}) = 0$. Then, since

$$v(\alpha_1) + \cdots + v(\alpha_h) + v(\alpha_{h+1}) = v(\alpha_1) + \cdots + v(\alpha_h)$$
$$\leq M(\alpha_1, \dots, \alpha_h) \leq M(\alpha_1, \dots, \alpha_{h+1}),$$

(4.5) is satisfied by $\alpha_1, \dots, \alpha_{h+1}$, too. As a consequence, if v is finite, then it is very simple to verify if v is satisfiable or not. As an example, if $\mathrm{Supp}(v) = \{\alpha_1, \dots, \alpha_n\}$ and these formulas are pairwise incompatible, then v is satisfiable if and only if $v(\alpha_1) + \cdots + v(\alpha_n) \leq 1$.

Also, the following stake interpretation of Proposition 4,3 is possible. Given a function $v : \mathbb{F} \to U$, consider a game such that

- a player can bet on the occurrences of the events described by the formulas in $\mathbb{F}$;
- for every $\alpha \in \mathbb{F}$, betting on α costs $v(\alpha)$;
- the bank pays a fixed stake of one unit.

It is also possible to bet on several different events; in that case, if a player wants to bet on the (not necessarily distinct) events $\alpha_1, \dots, \alpha_h$, then he has to pay $v(\alpha_1) + \cdots + v(\alpha_h)$. $M(\alpha_1, \dots, \alpha_h)$ represents the maximum amount of money a player betting on $\alpha_1, \dots, \alpha_h$ can win. We call any bet $\alpha_1, \dots, \alpha_h$ *player-reasonable* in which the player has the possibility of receiving at least the amount of money that he has disbursed. Thus, Proposition 4.3 says that v is satisfiable if and only if every bet in the related game is player-reasonable. (Obviously, the term "player-reasonable" refers to the player's point of view and not to the bank's point of view.) As an example, due to the zero rule, the betting function of the roulette game is not player-reasonable, and therefore it is unsatisfiable.

5. REFUTATIONS FOR PROBABILITY LOGIC

Proposition 4.3, together with Propositions 3.2 and 3.3, gives an effective way to compute the quantites $C(v)(\alpha)$ and $C^*(v)(\alpha)$. Also, by giving some suitable definitions, we are able to write two simple formulas. Namely, set $M(\alpha_1, \dots, \alpha_h / \alpha) = M(\alpha_1 \wedge \alpha, \dots, \alpha_h \wedge \alpha)$, that is,

$$M(\alpha_1, \dots, \alpha_h / \alpha) = \max\{k \in N | C^k(\alpha_1, \dots, \alpha_h) \text{ is compatible with } \alpha\}. \tag{5.1}$$

In terms of the betting interpretation, $M(\alpha_1, \dots, \alpha_h / \alpha)$ is the maximum amount of money a player betting on $\alpha_1, \dots, \alpha_h$ can win "given α." Also, the function $d(\alpha_1, \dots, \alpha_h / \alpha)$ defined by

$$d(\alpha_1, \dots, \alpha_h / \alpha) = M(\alpha_1, \dots, \alpha_h) - M(\alpha_1, \dots, \alpha_h / \alpha) \tag{5.2}$$

will be useful. The betting meaning of $d(\alpha_1,\dots,\alpha_h/\alpha)$ is obvious. Assume that v is satisfiable; we define the *degree of consistency* $\mathrm{Cons}(\alpha_1,\dots,\alpha_h/\alpha)$ *of* α *with* $\alpha_1,\dots,\alpha_h$ by setting $\mathrm{Cons}(\alpha_1,\dots,\alpha_h/\alpha) = 1$ if $d(\alpha_1,\dots,\alpha_h/\alpha) = 0$ and

$$\mathrm{Cons}(\alpha_1,\dots,\alpha_h/\alpha) = \frac{M(\alpha_1,\dots,\alpha_h) - [v(\alpha_1) + \cdots + v(\alpha_h)]}{d(\alpha_1,\dots,\alpha_h/\alpha)} \tag{5.3}$$

otherwise. The *degree of inconsistency* $Inc(\alpha_1,\dots,\alpha_h/\alpha)$ is defined by

$$\mathrm{Inc}(\alpha_1,\dots,\alpha_h/\alpha) = 1 - \mathrm{Cons}(\alpha_1,\dots,\alpha_h/\alpha).$$

Consequently, $\mathrm{Inc}(\alpha_1,\dots,\alpha_h/\alpha) = 0$ if $d(\alpha_1,\dots,\alpha_h/\alpha) = 0$, and

$$\mathrm{Inc}(\alpha_1,\dots,\alpha_h/\alpha) = \frac{v(\alpha_1) + \cdots + v(\alpha_h) - M(\alpha_1,\dots,\alpha_h/\alpha)}{d(\alpha_1,\dots,\alpha_h/\alpha)} \tag{5.4}$$

otherwise.

PROPOSITION 5.1 *Let α be a formula and v a satisfiable initial valuation. Then*

$$C(v)(\alpha) = \sup\{Inc(\alpha_1,\dots,\alpha_h/\sim\alpha) | \alpha_1,\dots,\alpha_h \in Supp(v)\}, \tag{5.5}$$

$$C(v)^*(\alpha) = \inf\{Cons(\alpha_1,\dots,\alpha_h/\alpha) | \alpha_1,\dots,\alpha_h \in Supp(v)\}. \tag{5.6}$$

Proof First observe that

$$d(\alpha_1,\dots,\alpha_h/\alpha) = \max\{s \in N | M(\alpha_1,\dots,\alpha_h,\alpha^S) = M(\alpha_1,\dots,\alpha_h)\}, \tag{5.7}$$

where $\alpha_1,\dots,\alpha_h,\alpha^S$ is the sequence obtained by adding the formula α to $\alpha_1,\dots,\alpha_h$ s times, namely the sequence $\alpha_1,\dots,\alpha_h,\alpha_{h+1},\dots,\alpha_{h+s}$, where $\alpha_{h+1} = \cdots = \alpha_{h+s} = \alpha$. Indeed, set $m = M(\alpha_1,\dots,\alpha_h)$; since

$$M(\alpha_1,\dots,\alpha_h,\alpha^S) = \max\{m, s + M(\alpha_1,\dots,\alpha_h/\alpha)\}, \tag{5.8}$$

from $s \le d(\alpha_1,\dots,\alpha_h/\alpha)$ it follows that

$$\begin{aligned} m &\le M(\alpha_1,\dots,\alpha_h,\alpha^S) \\ &\le \max\{m, d(\alpha_1,\dots,\alpha_h/\alpha) + M(\alpha_1,\dots,\alpha_h/\alpha)\} = m, \end{aligned}$$

and therefore $M(\alpha_1,\dots,\alpha_h) = M(\alpha_1,\dots,\alpha_h,\alpha^s)$. On the other hand, if

$s > d(\alpha_1, \dots, \alpha_h/\alpha)$, then

$$M(\alpha_1, \dots, \alpha_h, \alpha^s) \geq s + M(\alpha_1, \dots, \alpha_h/\alpha)$$
$$> d(\alpha_1, \dots, \alpha_h/\alpha) + M(\alpha_1, \dots, \alpha_h/\alpha) = m.$$

Now, recall that

$$\mathrm{Inc}(v, \alpha) = \sup\{\lambda \in U | v \cup \{\alpha\}^{1-\lambda} \text{ is not satisfiable}\}$$

and that, by Proposition 4.3, $v \cup \{\alpha\}^{1-\lambda}$ is not satisfiable if and only if $\alpha_1, \dots, \alpha_h \in \mathrm{Supp}(v)$ and $s \neq 0$ exist such that

$$v(\alpha) + \cdots + v(\alpha_h) + s \cdot (1 - \lambda) > M(\alpha_1, \dots, \alpha_h, \alpha^S). \tag{5.9}$$

Now, by (5.8),

$$v(\alpha_1) + \cdots + v(\alpha_h) + s \cdot (1 - \lambda) > s + M(\alpha_1, \dots, \alpha_h/\alpha),$$

or equivalently,

$$\lambda < \frac{v(\alpha_1) + \cdots + v(\alpha_h) - M(\alpha_1, \dots, \alpha_h/\alpha)}{s}.$$

From this inequality it follows that $v(\alpha_1) + \cdots + v(\alpha_h) > M(\alpha_1, \dots, \alpha_h/\alpha)$ and therefore, by the consistency of v, that $m > M(\alpha_1, \dots, \alpha_h/\alpha)$. So $d(\alpha_1, \dots, \alpha_h/\alpha) \neq 0$.

Now, it is not restrictive to assume that

$$s \geq d(\alpha_1, \dots, \alpha_h/\alpha). \tag{5.10}$$

Indeed, in case $s < d(\alpha_1, \dots, \alpha_h/\alpha)$, set $s' = d(\alpha_1, \dots, \alpha_h/\alpha)$. Then, since by (5.8)

$$M(\alpha_1, \dots, \alpha_h, \alpha^s) = M(\alpha_1, \dots, \alpha_h, \alpha^{s'}) = m,$$

we have that

$$v(\alpha_1) + \cdots + v(\alpha_h) + s' \cdot (1 - \lambda) > v(\alpha_1) + \cdots + v(\alpha_h) + s \cdot (1 - \lambda)$$
$$> M(\alpha_1, \dots, \alpha_h, \alpha^s)$$
$$= M(\alpha_1, \dots, \alpha_h, \alpha^{s'}),$$

and this means that s' satisfies both (5.9) and (5.10). Thus, $\lambda < \mathrm{Inc}(\alpha_1, \dots, \alpha_h/\alpha)$, and this proves that

$$\mathrm{Inc}(v, \alpha) \leq \sup\{\mathrm{Inc}(\alpha_1, \dots, \alpha_h/\alpha) | \alpha_1, \dots, \alpha_h \in \mathrm{Supp}(v)\}.$$

Conversely, let $\lambda < \sup\{\mathrm{Inc}(\alpha_1,\ldots,\alpha_h/\alpha) \mid \alpha_1,\ldots,\alpha_h \in \mathrm{Supp}(v)\}$. Then $\alpha_1,\ldots,\alpha_h$ exist such that $\lambda < \mathrm{Inc}(\alpha_1,\ldots,\alpha_h/\alpha)$. Then we have that $d(\alpha_1,\ldots,\alpha_h/\alpha) \neq 0$ and that

$$v(\alpha_1) + \cdots + v(\alpha_h) - M(\alpha_1,\ldots,\alpha_h/\alpha) > \lambda d(\alpha_1,\ldots,\alpha_h/\alpha).$$

By adding $M(\alpha_1,\ldots,\alpha_h)$ to both sides,

$$v(\alpha_1) + \cdots + v(\alpha_h) + (1-\lambda)d(\alpha_1,\ldots,\alpha_h/\alpha) > M(\alpha_1,\ldots,\alpha_h).$$

If we set $s = d(\alpha_1,\ldots,\alpha_h/\alpha)$, then since $M(\alpha_1,\ldots,\alpha_h) = M(\alpha_1,\ldots,\alpha_h,\alpha^s)$, we may rewrite this inequality as follows:

$$v(\alpha_1) + \cdots + v(\alpha_h) + s\cdot(1-\lambda) > M(\alpha_1,\ldots,\alpha_h,\alpha^s),$$

and $v \cup \{\alpha\}^{1-\lambda}$ is not satisfiable. This entails that $\lambda \leq \mathrm{Inc}(v,\alpha)$ and therefore that

$$\mathrm{Inc}(v,\alpha) \geq \sup\{\mathrm{Inc}(\alpha_1,\ldots,\alpha_h/\alpha) | \alpha_1,\ldots,\alpha_h \in \mathrm{Supp}(v)\}.$$

This concludes the proof of (5.5).

(5.6) is an immediate consequence of (5.5). ■

Notice that each interval $[\mathrm{Inc}(\alpha_1,\ldots,\alpha_h/\sim\alpha), \mathrm{Cons}(\alpha_1,\ldots,\alpha_h/\alpha)]$ gives an approximation of the unknown probability $p(\alpha)$, and this suggests that (5.5) and (5.6) are useful tools for the questions examined in [14]. It is possible to give a precise formulation of the effectiveness of the formulas (5.5) and (5.6). Indeed, in [2] and [3], suitable notions of decidability and of recursive enumerability for fuzzy subsets are proposed. Namely, if S is a set (with a coding), then we call a fuzzy subset s of S *recursively enumerable* if $s(x) = \lim_{n\to\infty} h(x,n)$ for every $x \in S$, where $h : \mathbb{F} \times N \to U$ is a computable map increasing with respect to n and with rational values. Equivalently, s is recursively enumerable if $s(x) = \sup\{h(x,n) \mid n \in N\}$ with $h(x,n)$ computable and with rational values. We say that s is *recursively coenumerable* if its complement is recursively enumerable, and that s is *decidable* if s is both recursively enumerable and recursively coenumerable. This means that s is decidable provided that two effectively computable maps $h : \mathbb{F} \times N \to U$ and $k : \mathbb{F} \times N \to U$ exist such that $s(x)$ is the limit of the sequence $[h(x,n), k(x,n)]$ of nested intervals.

In accordance with this terminology, it is immediate that if the initial valuation v is computable, then $C(v)$ is recursively enumerable and $C(v)^*$ is recursively coenumerable. If v is complete, that is, if $C(v)$ is a probability model, then $C(v)$ is a decidable fuzzy subset. In the terminology of a logician, every axiomatizable theory is recursively enumerable, and every axiomatizable, complete theory is decidable.

6. MULTIVALUED LOGIC

The interval U, equipped with a suitable set of operations, is the basis of several multivalued logics, for example Łukasiewicz's logics. Now, given a multivalued logic, the class $\mathcal{M}$ of all truth functional valuations defines a fuzzy semantics in a natural way. In the following we assume that

- negation is interpreted by setting $m(\sim a) = 1 - m(\alpha)$ for every $m \in \mathcal{M}$ and $\alpha \in \mathbb{F}$;
- all the connectives are interpreted by continuous functions;
- there are only a finite number of propositional variables $p_1, \dots, p_k$.

The first condition means that $\mathcal{M}$ is balanced. Then, an initial valuation $v : \mathbb{F} \to U$ defines, for every formula α, a constraint $[v(\alpha), v^*(\alpha)]$ of the unknown "actual" truth value $m(\alpha)$, and we may improve such constraints by considering the intervals $[C(v)(\alpha), C(v)^*(\alpha)]$. The second condition entails that $\mathcal{M}$ is closed with respect to the ultraproducts and therefore that $\mathcal{M}$ is s-compact. The last condition enables us to identify a model with a point $(\lambda_1, \dots, \lambda_k)$ in the space U^k.

Now, for every formula α, we denote by $f_\alpha : U^k \to U$ the polynomial function associated with α, and, as usual, we call two formulas α and β *logically equivalent* if $f_\alpha = f_\beta$. If $\gamma_1, \gamma_2, \dots$ is an enumeration of all the formulas, then for every formula α we have that $v \cup \{\alpha\}^\lambda$ is satisfiable if and only if the inequalities

$$\begin{aligned} f_{\gamma_1}(x_1, \dots, x_k) &\geq v(\gamma_1), \\ f_{\gamma_2}(x_1, \dots, x_k) &\geq v(\gamma_2), \\ &\vdots \end{aligned} \tag{6.1}$$

are satisfied by numbers $\lambda_1, \dots, \lambda_k$ in U such that

$$f_\alpha(\lambda_1, \dots, \lambda_k) \geq \lambda.$$

Let $E(v)$ be the class of solutions of (6.1). Then, since $C^*(v)(\alpha) = \mathrm{Cons}(v, \alpha)$, we have

$$C^*(v)(\alpha) = \sup\{f_\alpha(\lambda_1, \dots, \lambda_k) | (\lambda_1, \dots, \lambda_k) \in E(v)\}. \tag{6.2}$$

Likewise, since $C(v)(\alpha) = 1 - C^*(v)(\sim a)$ and $f_{\sim \alpha}(\lambda_1, \dots, \lambda_k) = 1 - f_\alpha(\lambda_1, \dots, \lambda_k)$,

$$C(v)(\alpha) = \inf\{f_\alpha(\lambda_1, \dots, \lambda_k) | (\lambda_1, \dots, \lambda_k) \in E(v)\}. \tag{6.3}$$

Since the logical connectives are interpreted by continuous functions, f_γ is a continuous function for any formula γ. Then $E(v)$ is compact, and, in accordance with Proposition 2.4, if v is satisfiable, $C(v)(\alpha)$ and $C^*(v)(\alpha)$ are the minimum and the maximum of f_α in $E(v)$.

To give an example, examine the very elementary multivalued logic whose connectives $\wedge$, $\vee$, and $\sim$ are interpreted by the minimum, the maximum, and the function $1 - x$, respectively. In other words, assume that $\mathscr{M}$ is the class of fuzzy subsets m of $\mathbb{F}$ such that

$$m(\alpha_1 \vee \alpha_2)) = \max\{m(\alpha_1), m(\alpha_2)\},$$

$$m(\alpha_1 \wedge \alpha_2)) = \min\{m(\alpha_1), m(\alpha_2)\},$$

$$m(\sim \alpha) = 1 - m(\alpha).$$

Notice that $\mathscr{M}$ is an extension of the classical semantics, since every classical valuation is an element of $\mathscr{M}$.

PROPOSITION 6.1 *Given an initial valuation* v, *the theory* $C(v)$ *does not preserve either the negation or the disjunction in general. Nevertheless, for every* α, $\beta \in \mathbb{F}$

$$C(v)(\alpha \wedge \beta) = C(v)(\alpha) \wedge C(v)(\beta). \tag{6.4}$$

Proof Let v be the empty fuzzy subset of $\mathbb{F}$, that is, the map constantly equal to zero, and α a formula such that either α nor $\sim \alpha$ is a classical tautology. Then, since classical valuations m_1 and m_2 exist such that $m_1(\alpha) = 0$, $m_2(\sim \alpha) = 0$, we have that $C(v)(\alpha) = C(v)(\sim \alpha) = 0$ and therefore that $C(v)(\sim \alpha) \neq 1 - C(v)(\alpha)$. Also, while it is immediate that $C(v)(\alpha \vee \sim \alpha) = \frac{1}{2}$, we have that $C(v)(\alpha) \vee C(v)(\sim \alpha) = 0$.

To prove (6.4), let v be any initial valuation and observe that

$$\begin{aligned} C(v)(\alpha \wedge \beta) &= \inf\{m(\alpha \wedge \beta) | m \in \mathscr{M}, m \supseteq v\} \\ &= \inf\{m(\alpha) \wedge m(\beta) | m \in \mathscr{M}, m \supseteq v\} \\ &= \inf\{m(\alpha) | m \in \mathscr{M}, m \supseteq v\} \wedge \inf\{m(\beta) | m \in \mathscr{M}, m \supseteq v\} \\ &= C(v)(\alpha) \wedge C(v)(\beta). \end{aligned} \qquad \blacksquare$$

(6.4) enables us to simplify the computation of $C(v)$ and $C(v)^*$. Indeed, due to the properties of a Morgan algebra, for the multivalued calculus under consideration the conjunctive normal form theorem holds. Namely, recall that a *literal* is either a proportional variable p_i or its negation $\sim p_i$, and that a *clause* is a disjunction of literals. Then every formula is equivalent to a conjunction of clauses, and there are only a finite number of equivalence classes modulo the logical equivalence relation. Now, assume that γ_i is reduced to a conjunctive normal form $\delta_1^i \wedge \cdots \wedge \delta_k^i$, where $\delta_1^i, \ldots, \delta_k^i$ are clauses. Then, since

$$f_{\gamma_i}(\lambda_1, \ldots, \lambda_k) = f_{\delta_1^i}(\lambda_1, \ldots, \lambda_k) \wedge \cdots \wedge f_{\delta_k^i}(\lambda_1, \ldots, \lambda_k),$$

each condition $f_{\gamma_i}(\lambda_1, \ldots, \lambda_k) \geq v(\gamma_i)$ comes down to the conditions $f_{\delta_j^i}(\lambda_1, \ldots, \lambda_k) \geq v(\gamma_i)$, $j = 1, \ldots, k$, and therefore to a finite set of conditions of the type

$$\lambda_1^{j_1} \vee \cdots \vee \lambda_k^{j_k} \geq v(\gamma_i),$$

where $j_1, \ldots, j_k \in \{-1, 1\}$ and where we set $\lambda^1 = \lambda$ and $\lambda^{-1} = 1 - \lambda$. As a consequence, $E(v)$ is defined by a finite system of formulas of the type

$$\lambda_1^{c(1,j)} \vee \cdots \vee \lambda_k^{c(k,j)} \geq \mu(j) \qquad (j = 1, \ldots, r), \tag{6.5}$$

where $c(i, j) \in \{-1, 1\}$. Also, since we may write each formula α in a conjunctive normal form, then by (6.4) $C(v)(\alpha)$ is the minimum of a finite set of numbers like $C(v)(p_1^{i_1} \vee \cdots \vee p_k^{i_k})$. Thus, the whole computation becomes a simple one like

$$\min\{\lambda_1^{i_1} \vee \cdots \vee \lambda_k^{i_k} | \lambda_1^{c(1,j)} \vee \cdots \vee \lambda_k^{c(k,j)} \geq \mu(j), j = 1, \ldots, r\}.$$

Analogous arguments hold for $C(v)^*(\alpha)$.

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